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Existence of solutions for a certain differential inclusion of third order

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Abstract

The existence of solutions of a boundary value problem for a third order differential inclusion is investigated. New results are obtained by using suitable fixed point theorems when the right hand side has convex or non convex values.

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1 Introduction

This paper is concerned with the following boundary value problem

$$x''' + k^2 x' \in F(t,x), \quad a.e.\ ([-1,1]), \quad x(-1) = x(1) = x'(1) = 0, \tag{1.1}$$

where $F(.,.) : [-1,1] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $k \in [-\pi, \pi]$.

Problem (1.1) occurs in hydrodynamic and viscoelastic plates theory. For the motivation of the study of this class of problem we refer to [1] and references therein.

The present paper is motivated by a recent paper of Bartuzel and Fryszkowski ([1]), where it is considered problem (1.1) and a version of the Filippov lemma for this problem is provided. The aim of our paper is to present two other existence results for problem (1.1). Our results are essentially based on a nonlinear alternative of Leray-Schauder type and on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values. The methods used are standard, however their exposition in the framework of problem (1.1) is new.

We note that two other existence results for problem (1.1) obtained by the application of the set-valued contraction principle due to Covitz and Nadler jr. may be found in our previous paper [3].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space with the corresponding norm $|.|$ and let $I \subset \mathbf{R}$ be a compact interval. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0, 1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_I |x(t)| \mathrm{d}t$.

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

Consider $T : X \to \mathcal{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for $T(.)$ if $x \in T(x)$. $T(.)$ is said to be bounded on bounded sets if $T(B) := \cup_{x \in B} T(x)$ is a bounded subset of X for all bounded sets B in X. $T(.)$ is said to be compact if $T(B)$ is relatively compact for any bounded sets B in X. $T(.)$ is said to be totally compact if $\overline{T(X)}$ is a compact subset of

X. $T(.)$ is said to be upper semicontinuous if for any open set $D \subset X$, the set $\{x \in X; T(x) \subset D\}$ is open in X. $T(.)$ is called completely continuous if it is upper semicontinuous and totally bounded on X.

It is well known that a compact set-valued map $T(.)$ with nonempty compact values is upper semicontinuous if and only if $T(.)$ has a closed graph.

We recall the following nonlinear alternative of Leray-Schauder type and its consequences.

Theorem 2.1. ([6]) *Let D and $\overline{D}$ be the open and closed subsets in a normed linear space X such that $0 \in D$ and let $T : \overline{D} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.3. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

i) the equation $x = T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

We recall that a multifunction $T(.) : X \to \mathcal{P}(X)$ is said to be lower semicontinuous if for any closed subset $C \subset X$, the subset $\{s \in X; G(s) \subset C\}$ is closed.

If $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map with compact values and $x(.) \in C(I, \mathbf{R})$ we define

$$S_F(x) := \{f \in L^1(I, \mathbf{R}); \quad f(t) \in F(t, x(t)) \quad a.e.\ (I)\}.$$

We say that $F(.,.)$ is of *lower semicontinuous type* if $S_F(.)$ is lower semicontinuous with closed and decomposable values.

Theorem 2.4. ([2]) *Let S be a separable metric space and $G(.) : S \to \mathcal{P}(L^1(I, \mathbf{R}))$ be a lower semicontinuous set-valued map with closed decomposable values.*

Then $G(.)$ has a continuous selection (i.e., there exists a continuous mapping $g(.) : S \to L^1(I, \mathbf{R})$ such that $g(s) \in G(s) \quad \forall s \in S$).

A set-valued map $G : I \to \mathcal{P}(\mathbf{R})$ with nonempty compact convex values is said to be *measurable* if for any $x \in \mathbf{R}$ the function $t \to d(x, G(t))$ is measurable.

A set-valued map $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is said to be *Carathéodory* if $t \to F(t, x)$ is measurable for any $x \in \mathbf{R}$ and $x \to F(t, x)$ is upper semicontinuous for almost all $t \in I$.

$F(.,.)$ is said to be L^1-*Carathéodory* if for any $l > 0$ there exists $h_l(.) \in L^1(I, \mathbf{R})$ such that $\sup\{|v|; v \in F(t, x)\} \leq h_l(t)$ a.e. (I), $\forall x \in \overline{B_l(0)}$.

Theorem 2.5. ([5]) *Let X be a Banach space, let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a L^1-Carathéodory set-valued map with $S_F \neq \emptyset$ and let $\Gamma : L^1(I, X) \to C(I, X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_F : C(I, X) \to \mathcal{P}(C(I, X))$ defined by

$$(\Gamma \circ S_F)(x) = \Gamma(S_F(x))$$

has compact convex values and has a closed graph in $C(I, X) \times C(I, X)$.

Note that if $dim X < \infty$, and $F(.,.)$ is as in Theorem 2.5, then $S_F(x) \neq \emptyset$ for any $x(.) \in C(I, X)$ (e.g., [5]).

In what follows $I = [-1, 1]$ and let $AC^2(I, \mathbf{R})$ be the space of two times differentiable functions $x(.) : I \to \mathbf{R}$ whose second derivative exists and is absolutely continuous on I. On $AC^2(I, \mathbf{R})$ we consider the norm $|.|_C$.

By a solution of problem (1.1) we mean a function $x(.) \in AC^2(I, \mathbf{R})$ if there exists a function $f(.) \in L^1(I, \mathbf{R})$ with $f(t) \in F(t, x(t))$, *a.e.* (I) such that $x'''(t) + k^2 x'(t) = f(t)$ *a.e.* (I) and $x(-1) = x(1) = x'(1) = 0$.

Lemma 2.6. ([1]) *If $f(.) : [-1, 1] \to \mathbf{R}$ is an integrable function and $k \in [-\pi, \pi]$ then the equation*

$$x''' + k^2 x' = f(t) \quad a.e. \ (I),$$

with the boundary conditions $x(-1) = x(1) = x'(1) = 0$ *has a unique solution given by*

$$x(t) = \int_{-1}^{1} G(t,s) f(s) ds,$$

where $G(.,.)$ *is the associated Green function. Namely,*

$$G(t,x) = \begin{cases} \frac{(1-cosk(1+x))(1-cosk(1-t))}{k^2(1-cos2k)} & \text{if} \quad -1 \le x \le t \le 1, \\ \frac{(1-cosk(1+x))(1-cosk(1-t))-(1-cosk(x-t))(1-cos2k)}{k^2(1-cos2k)} & \text{if } -1 \le t \le x \le 1. \end{cases}$$

Moreover, if $k \neq 0$

$$0 \le G(t,x) \le G_0 := \frac{k^2(5\sqrt{5}-11)}{sin^2 k} \quad \forall\ (t,x) \in I \times \mathbf{R}.$$

3 The main results

We are able now to present the existence results for problem (1.1). We consider first the case when $F(.,.)$ is convex valued.

Hypothesis 3.1. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.

ii) There exist $\varphi(.) \in L^1(I, \mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$\sup\{|v|; \quad v \in F(t,x)\} \le \varphi(t)\psi(|x|) \quad a.e.\ (I), \quad \forall x \in \mathbf{R}.$$

Theorem 3.2. *Assume that Hypothesis 3.1 is satisfied and there exists* $r > 0$ *such that*

$$r > G_0 |\varphi|_1 \psi(r). \tag{3.1}$$

Then problem (1.1) has at least one solution $x(.)$ *such that* $|x(.)|_C < r$.

Proof. Let $X = AC^2(I, \mathbf{R})$ and consider $r > 0$ as in (3.1). It is obvious that the existence of solutions to problem (1.1) reduces to the existence of the solutions of the integral inclusion

$$x(t) \in \int_{-1}^{1} G(t,s) F(s, x(s)) ds, \quad t \in I. \tag{3.2}$$

Consider the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(AC^2(I,\mathbf{R}))$ defined by

$$T(x) := \{v(.) \in AC^2(I,\mathbf{R});\ v(t) := \int_{-1}^{1} G(t,s)f(s)ds,\ f \in \overline{S_F(x)}\}. \quad (3.3)$$

We show that $T(.)$ satisfies the hypotheses of Corollary 2.2.

First, we show that $T(x) \subset AC^2(I,\mathbf{R})$ is convex for any $x \in AC^2(I,\mathbf{R})$.

If $v_1, v_2 \in T(x)$ then there exist $f_1, f_2 \in S_F(x)$ such that for any $t \in I$ one has

$$v_i(t) = \int_{-1}^{1} G(t,s)f_i(s)ds, \quad i = 1, 2.$$

Let $0 \le \alpha \le 1$. Then for any $t \in I$ we have

$$(\alpha v_1 + (1-\alpha)v_2)(t) = \int_{-1}^{1} G(t,s)[\alpha f_1(s) + (1-\alpha)f_2(s)]ds.$$

The values of $F(.,.)$ are convex, thus $S_F(x)$ is a convex set and hence $\alpha h_1 + (1-\alpha)h_2 \in T(x)$.

Secondly, we show that $T(.)$ is bounded on bounded sets of $AC^2(I,\mathbf{R})$.

Let $B \subset AC^2(I,\mathbf{R})$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \le m\ \forall x \in B$.

If $v \in T(x)$ there exists $f \in S_F(x)$ such that $v(t) = \int_{-1}^{1} G(t,s)f(s)ds$. One may write for any $t \in I$

$$|v(t)| \le \int_{-1}^{1} |G(t,s)|.|f(s)|ds \le \int_{-1}^{1} |G(t,s)|\varphi(s)\psi(|x(t)|)ds$$

and therefore

$$|v|_C \le G_0|\varphi|_1\psi(m) \quad \forall v \in T(x),$$

i.e., $T(B)$ is bounded.

We show next that $T(.)$ maps bounded sets into equi-continuous sets.

Let $B \subset AC^2(I,\mathbf{R})$ be a bounded set as before and $v \in T(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = \int_{-1}^{1} G(t,s)f(s)ds$. Then for any $t, \tau \in I$ we have

$$|v(t) - v(\tau)| \le |\int_{-1}^{1} G(t,s)f(s)ds - \int_{-1}^{1} G(\tau,s)f(s)ds| \le$$

$$\int_{-1}^{1} |G(t,s) - G(\tau,s)|.|f(s)|ds \le \int_{-1}^{1} |G(t,s) - G(\tau,s)|\varphi(s)\psi(m)ds.$$

It follows that $|v(t) - v(\tau)| \to 0$ as $t \to \tau$. Therefore, $T(B)$ is an equicontinuous set in $AC^2(I, \mathbf{R})$.

We apply now Arzela-Ascoli's theorem we deduce that $T(.)$ is completely continuous on $AC^2(I, \mathbf{R})$.

In the next step of the proof we prove that $T(.)$ has a closed graph.

Let $x_n \in AC^2(I, \mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in T(x_n)$ $\forall n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in T(x^*)$.

Since $v_n \in T(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = \int_{-1}^{1} G(t,s) f_n(s) ds$.

Define $\Gamma : L^1(I, \mathbf{R}) \to AC^2(I, \mathbf{R})$ by $(\Gamma(f))(t) := \int_{-1}^{1} G(t,s) f(s) ds$. One has $\max_{t \in I} |v_n(t) - v^*(t)| = |v_n(.) - v^*(.)|_C \to 0$ as $n \to \infty$

We apply Theorem 2.5 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) = \int_{-1}^{1} G(t,s) f^*(s) ds$.

Therefore, $T(.)$ is upper semicontinuous and compact on $\overline{B_r(0)}$. We apply Corollary 2.2 to deduce that either i) the inclusion $x \in T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Assume that ii) is true. With the same arguments as in the second step of our proof we get $r = |x(.)|_C \le G_0 |\varphi|_1 \psi(r)$ which contradicts (3.1). Hence only i) is valid and theorem is proved.

We consider now the case when $F(.,.)$ is not necessarily convex valued. Our existence result in this case is based on the Leray-Schauder alternative for single valued maps and on Bressan Colombo selection theorem.

Hypothesis 3.3. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has compact values, $F(.,.)$ is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable and $x \to F(t,x)$ is lower semicontinuous for almost all $t \in I$.

ii) There exist $\varphi(.) \in L^1(I, \mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$\sup\{|v|; \quad v \in F(t,x)\} \le \varphi(t)\psi(|x|) \quad a.e.\ (I), \quad \forall x \in \mathbf{R}.$$

Theorem 3.4. *Assume that Hypothesis 3.3 is satisfied and there exists $r > 0$ such that Condition (3.1) is satisfied.*

Then problem (1.1) has at least one solution on I.

Proof. We note first that if Hypothesis 3.3 is satisfied then $F(.,.)$ is of lower semicontinuous type (e.g., [4]). Therefore, we apply Theorem 2.4 to deduce that there exists $f(.) : AC^2(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ such that $f(x) \in S_F(x)$ $\forall x \in AC^2(I, \mathbf{R})$.

We consider the corresponding problem

$$x(t) = \int_{-1}^{1} G(t,s) f(x(s)) ds, \quad t \in I \tag{3.4}$$

in the space $X = AC^2(I, \mathbf{R})$. It is clear that if $x(.) \in AC^2(I, \mathbf{R})$ is a solution of the problem (3.4) then $x(.)$ is a solution to problem (1.1).

Let $r > 0$ that satisfies condition (3.1) and define the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(AC^2(I, \mathbf{R}))$ by

$$(T(x))(t) := \int_{-1}^{1} G(t,s) f(x(s)) ds.$$

Obviously, the integral equation (3.4) is equivalent with the operator equation

$$x(t) = (T(x))(t), \quad t \in I. \tag{3.5}$$

It remains to show that $T(.)$ satisfies the hypotheses of Corollary 2.3.

We show that $T(.)$ is continuous on $\overline{B_r(0)}$. From Hypotheses 3.3. ii) we have

$$|f(x(t))| \leq \varphi(t) \psi(|x(t)|) \quad a.e.\ (I)$$

for all $x(.) \in AC^2(I, \mathbf{R})$. Let $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then

$$|f(x_n(t))| \leq \varphi(t) \psi(r) \quad a.e.\ (I).$$

From Lebesgue's dominated convergence theorem and the continuity of $f(.)$ we obtain, for all $t \in I$

$$\lim_{n \to \infty} (T(x_n))(t) = \int_{-1}^{1} G(t,s) f(x_n(s)) ds = \int_{-1}^{1} G(t,s) f(x(s)) ds = (T(x))(t),$$

i.e., $T(.)$ is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.2 with corresponding modifications it follows that $T(.)$ is compact on $\overline{B_r(0)}$. We apply Corollary 2.3 and we find that either i) the equation $x = T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

As in the proof of Theorem 3.2 if the statement ii) holds true, then we obtain a contradiction to (3.1). Thus only the statement i) is true and problem (1.1) has a solution $x(.) \in AC^2(I, \mathbf{R})$ with $|x(.)|_C < r$.

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